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Scaling Integration Method for Singular Integrals and Its Application to BEM

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A new, accurate, and efficient algorithm is developed for the numerical evaluation of a class of singular integrals which arise in the BEM (boundary element method) analyses. The algorithm is based on an idea of scaling integration that utilizes a proportional relationship inherent in the numerical computation of the singular integrals over flat triangular elements. The present algorithm (scaling integration method) avoids the direct numerical evaluation of the integrand in the neighborhood of singularities. It is shown and demonstrated that the method is simpler, faster, and more accurate than standard Gaussian integration algorithms. An application of the method to the BEM analysis is also described and the results are favorably compared with the exact solution. © 1989 Academic Press, Inc.

1. INTRODUCTION

The boundary element method (BEM) for linear elastic problems is well known and well documented elsewhere, see, for example, Brebbia [1], Banerjee and Butterfield [2], and Tanaka [3]. The main advantage of BEM is the numerical efficiency; since it avoids discretizing the internal field or body, only the boundary of the body requires discretization. The method employs a fundamental solution or a Green's function of the relevant differential equation to reformulate the problem as a boundary value problem in terms of an integral equation. In the BEM formulation, the singularities of the fundamental solution are generally located on the boundary of the problem. This presents some numerical difficulties, and special care is usually exercised to evaluate singular integrals in the vicinity of the singularities.

Various methods have been proposed to cope with the difficulties in numerical integration of a singular integrand. On this subject, the readers are referred to Lachat and Watson [4], Critescu and Loubignoc [5], and Hayami and Brebbia [6]. Lean and Wexler [7] proposed a method to adjust weights of the Gaussian quadrature scheme, which takes into account of the effects of singularities. Aliabadi and Hall [8] utilized the Taylor expansion of integrand so that the singular terms can be integrated in closed form. Telles [9] developed an algorithm which yields

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locations of the best sampling points with minimum integration error when singularity is either on or inside of the boundary surface.

Another approach is to ensure that the singularities and the boundary nodes are not located on the same boundary. Oliveira [10] suggested moving the singularities of the free-space Green's function onto the auxiliary boundary. Han and Olson [11] proposed the adaptive BEM where the singularities are positioned on an auxiliary boundary which is allowed to move. Although the method is able to reduce the number of singularities, it involves a nonlinear method and requires extra iteration in computing the singularity locations.

An alternative may be to devise an algorithm which avoids the direct evaluation of singular integrals. The Green's function in BEM analyses usually has the r-inverse singularity, where r represents the distance between the boundary nodes. In this paper, a new, fast, and accurate algorithm for this class of singular integration problems is presented and the effectiveness of the algorithm is confirmed by applying it to the simple example in BEM analysis. The present algorithm (scaling integration method) provides a means to avoid numerical evaluation of the integrand in the neighborhood of singularities. The method utilizes the inherent (analytical) scaling relationship of the integration value with respect to the flat triangular area of integration. The derivation of the scaling integration method is described in Section 2. The efficiency and accuracy of the method are demonstrated in Section 3 by applying it to a simple BEM problem whose analytical solution is known. In Sections 4 and 5, a general extension of the present method and conclusions are given, respectively.

2. Scaling Integration Method

2.1. *Problem Formulation*

Let us consider the numerical evaluation of the integral of a certain function f over the finite triangular surface area S. We assume that f is the function with 3-dimensional spherical coordinates (r, θ, φ), and that f can be decomposed into the r-inverse part $1/r^{\beta}$ $(\beta < 2)$ and angular part $g(\theta, \varphi)$. It is further assumed that one of the vertexes of $\Delta ABC = S$ is located at the origin of the coordinate. The geometry of the problem is shown in Fig. 1. Then this integration, naturally a singular integration problem for $0 < \beta < 2$, can be written as follows:

$$\int_S f\, dS = \int_S \frac{g(\theta, \varphi)}{r^{\beta}}\, dS. \tag{1}$$

The assumptions on f and S may impose rather strong restrictions, but Green's functions associated with many important physical problems can be decomposed into the present form using the spherical coordinate. In particular, it is noted that

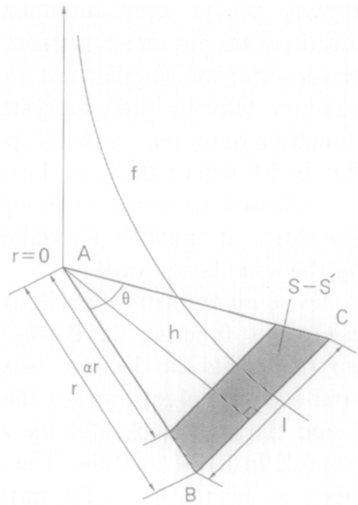

FIG. 1. Geometry of the singular integration problem.

the computation of the diagonal elements of the coefficient matrices in BEM analyses can be formulated in the present singular integral form (1) [1, 2].

Usually, this class of numerical integration problems is solved utilizing the Newtonian or Gaussian type of numerical algorithms [12]. In the conventional BEM algorithms, the boundary element mesh should be such that many sampling points are concentrated in the neighborhood of the singularities to guarantee the numerical accuracy; this implies that the computational cost is often prohibitively expensive. Numerical results, however, are generally poor due to the computational difficulties in evaluating f around the singular point A (where $r = 0$). In particular, the accuracy of computed derivatives on the boundary is often insufficiently poor.

To circumvent this difficulty, we propose to utilize a scaling or proportional relationship of the integral value with respect to the integration area S, which typically represents the triangular boundary element. The idea of the scaling integration method is illustrated in the subsequent development.

2.2. *Scaling Relationship*

If we slightly modify the domain of integration S to S', which is scaled (down) by the factor α (<1), then we obtain the scaling relationship

$$\int_S f\,dS: \quad \int_{S'} f\,dS' = 1: \quad \alpha^{(2-\beta)}, \tag{2}$$

where we have used the facts that the ratio of the triangular areas, S and S', is proportional to α^2 and that f is linear to $\alpha^{-\beta}$. Then, utilizing Eq. (2), the numerical integration of Eq. (1) can be performed without sampling the grid points adjacent to the singular point A as

$$
\begin{aligned}
\int f\,dS = I &= \frac{\int f\,dS - \int f\,dS'}{1-\alpha^{(2-\beta)}} \\
&= \frac{\int_{S-S'} f\,dt}{1-\alpha^{(2-\beta)}} \\
&\simeq \sum_{i=1}^{N} \frac{f_i\{(1-\alpha)+\alpha(1-\alpha)\}(hl/2N)}{1-\alpha^{(2-\beta)}} \\
&= \sum_{i=1}^{N} f_i \frac{hl}{2N} \frac{1-\alpha^2}{1-\alpha^{(2-\beta)}},
\end{aligned}
\tag{3}
$$

where we have divided the area $S-S'$ into the trapezoidal elements t_i ($i=1, 2, \ldots, N$), and f_i denotes the value of f evaluated at t_i; l is the length of **BC**, and h is the height of ΔABC (see Fig. 1). The domain of integration, $S-S'=\sum_{i=1}^{N} t_i$, obviously contains no singular point. Thus, the formulation (3) is very effective in the sense that it completely avoids the numerical evaluation of f around the singular point A.

When α is close to 1, the scaling integration method becomes simpler and more accurate. In the limit $\alpha \to 1$, the transformation (3) can be further approximated as

$$
\begin{aligned}
I &= \sum_{i=1}^{N} f_i \frac{hl}{2N} \frac{2}{(2-\beta)} \\
&= \sum_{i=1}^{N} f_i \frac{2S}{N(2-\beta)}.
\end{aligned}
\tag{4}
$$

The formulation (4) implies that the numerical integration can be performed by just multiplying a factor $2S/(2-\beta)$ to the mean value of f averaged over the boundary line elements along the edge **BC** of ΔABC.

It is clear that the numerical error of the present scaling integration method is a monotone decreasing function of the scaling factor α for the range of values, $0<\alpha\leqslant 1$. Hence, in the subsequent development, we will set $\alpha=1$. We may note that such a limit usually exists for the class of singular integration problems treated in this paper.

2.3. *Analytical Consideration*

The derivation of the scaling integration method can be demonstrated analytically by transforming the original surface integral to the line integral as

$$
\begin{aligned}
I &= \int_S \frac{g(\theta, \varphi)}{r'^{\beta}}\, dS \qquad (\beta < 2) \\
&= \int_{\theta_1}^{\theta_2} \int_0^{r(\theta)} \frac{g(\theta, \varphi)}{r'^{\beta}}\, r'\, dr'\, d\theta \\
&= \frac{1}{2-\beta} \int_{\theta_1}^{\theta_2} \frac{g(\theta, \varphi)}{r^{\beta-2}}\, d\theta \\
&= \frac{1}{2-\beta} \int_{y_1}^{y_2} \frac{g(y, \varphi)}{r^{\beta-2}} \frac{h}{r^2}\, dy \\
&= \frac{h}{2-\beta} \int_{y_1}^{y_2} \frac{g(y, \varphi)}{r^{\beta}}\, dy,
\end{aligned} \tag{5}
$$

where $dS = r'\, dr'\, d\theta$, $y = h \tan(\theta)$, $\arg(\mathbf{AB}) = \theta_1$, and $\arg(\mathbf{AC}) = \theta_2$. Considering $h \int_{y_1}^{y_2} dy = 2S$, this leads to the identical result to the previous formulation (4) if the integration over the edge **BC** is performed numerically along the boundary line elements.

The characteristics of the present scaling integration method may be summarized as follows:

1. It avoids numerical evaluation of the integrand in the neighborhood of the singular point,
2. it only needs to perform the numerical integration over boundary line elements,
3. it is not needed to apply any elaborate coordinate transformation using trigonometric functions, polynomial functions, etc., and,
4. it is, hence, able to provide a more accurate result with a smaller number of sampling points (i.e., less computation time) than the conventional methods.

It is noted that the present method can be applied only to the flat triangular elements. However, the algorithm may also be effectively applied to most of the smooth curved surfaces by locally subdividing and flattening the boundary surface.

3. Numerical Applications

3.1. *Simple Test Examples*

We compare the present scaling integration method with the conventional midpoint rule, Gauss rule, and exact solutions for the integration of the following two functions:

$$
f_1 = \frac{1}{r} \tag{6}
$$

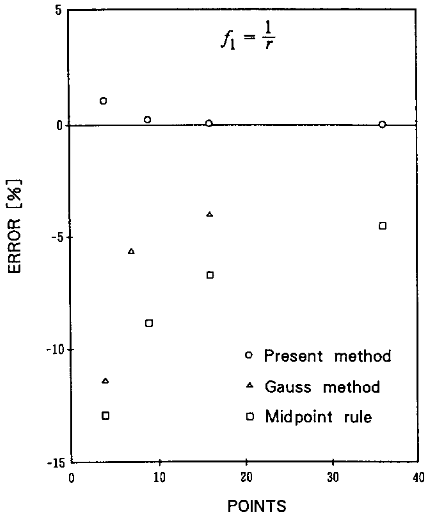

FIG. 2. Numerical accuracy for $f_1 = 1/r$.

and

$$f_2 = \frac{\cos\theta}{r} \tag{7}$$

The domain of integration is the triangular area with the vertexes (0, 0, 0), (0, 1, 0), and (1, 1, 0). In the numerical experiments, we vary the number of sampling points using the IBM Fortran software package with double precision arithmetic. It is reminded that we set $\alpha = 1$ for the present method.

Figures 2 and 3 show the results for two different functions, f_1 and f_2, respectively, where the numerical error is plotted as a function of the number of sampling points. Naturally, the numerical error is a decreasing function of the number of sampling points for each method. The numerical error of the present method is less than 2% for just sampling four points, while that of the conventional methods is greater than 4% using many more points than the scaling integration method. Since the computation time is proportional to (a certain power of) the number of sampling points, the numerical advantage of the present method is obvious.

3.2. *Application to the Boundary Element Method*

The following Navier equation in 3-dimensional space is considered to be solved employing the BEM technique,

$$L_{ij} u_j = 0, \tag{8}$$

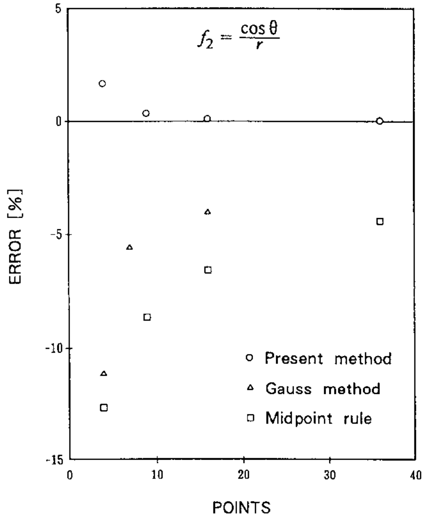

FIG. 3. Numerical accuracy for $f_2 = \cos\theta/r$.

where u_j is the displacement vector which is a function of the space coordinates x_i $(i = 1, 2, 3)$ and where index notation, with a summation denoted by repeated indices, is used. The partial differential operator L_{ij} is defined by

$$L_{ij} = \mu\,\delta_{ij}\,\Delta + \frac{\mu}{(1-2\nu)}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{9}$$

where δ_{ij} is the Kronecker's delta, Δ is the Laplacian operator, μ and ν are suitable constants.

Kelvin's solutions to Eq. (8), u_{ij}^* and p_{ij}^*, take the form

$$u_{ij}^* = \frac{1}{16\pi\mu(1-\nu)}\left(\frac{1}{r}\right)[(3-4\nu)\,\delta_{ij} + r,_i\,r,_j] \tag{10}$$

$$p_{ij}^* = \frac{(2\nu-1)}{8\pi(1-\nu)}\left(\frac{1}{r^2}\right)\left[\frac{\partial r}{\partial n}\left\{\delta_{ij} + \frac{3}{(1-2\nu)}r,_i\,r,_j\right\} + r,_j\,n_i - r,_i\,n_j\right], \tag{11}$$

where r and $r,_i$ denote the distance between the boundary and internal field points and its gradients $(i = 1, 2, 3)$, respectively, and n is the unit normal vector. It is clear that u_{ij}^* and p_{ij}^* are proportional to $1/\alpha$ and $1/\alpha^2$, respectively, if r is scaled by α (r is transformed to αr).

At first inspection of Eq. (11), the direct application of the present method to p_{ij}^*

seems to be not feasible. In the BEM formulation, however, the integrand becomes a product of a suitable shape (interpolation) function and p_{ij}^*, which enables the execution of the numerical integration, perfectly tractable. Therefore, by choosing the appropriate shape functions, the scaling integration method can be effectively applied to the BEM analysis.

The boundary integral equations for the boundary and field points, respectively, are written as

$$c_{ij}u_j + \int_S p_{ij}^* u_j \, dS = \int_S u_{ij}^* p_j \, dS \tag{12}$$

$$u_i + \int_S p_{ij}^* u_j \, dS = \int_S u_{ij}^* p_j \, dS, \tag{13}$$

where $c_{ij} = \frac{1}{2}\delta_{ij}$ for smooth surface and p_j denotes the traction vector. Usually, the surface integral which appears in Eqs. (12) and (13) is performed dividing the subject surface boundary into sufficiently well-discretized boundary elements. Note that the BEM formulation, in general, requires equal numbers of boundary nodes and singularities.

We consider here the cubic domain in which the numerical solution is obtained for Eqs. (12) and (13). The boundary conditions and the boundary (linear triangular) element mesh system are depicted in Fig. 4. If an element includes the origin, the scaling integration method is used as formulated in Section 2. Otherwise, a 5th-order (7 points) Gaussian quadrature scheme is used. The results of the

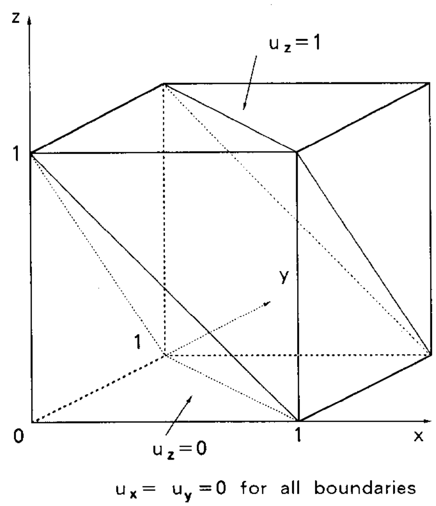

FIG. 4. Problem geometry for the BEM analysis.

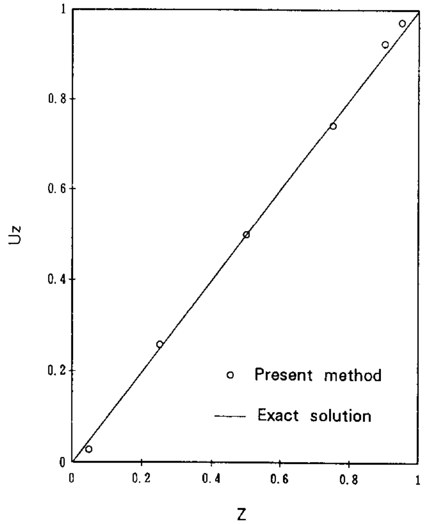

FIG. 5. Comparison of numerical and analytical solutions.

present BEM analysis and the analytical solution are compared for u_z in Fig. 5. The results indicate an excellent agreement of the numerical solution to the analytical solution. It is important to note that the value of u_z near the corner of the cubic domain is obtained with good accuracy; this can be a major numerical advantage of the present method.

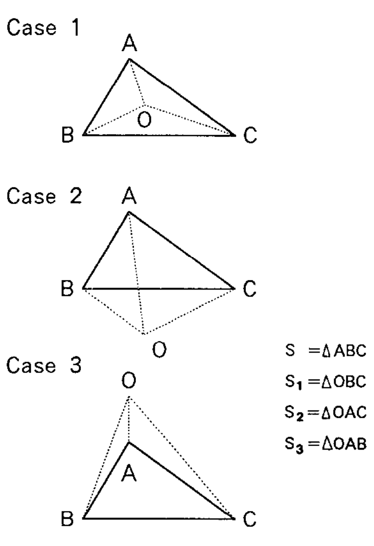

FIG. 6. General cases for relative location of the origin.

4. General Extension

In Section 2, it is assumed that the origin of the coordinate should be identical to one of the three vertexes of $\Delta ABC = S$. This restriction can be relaxed for general cases and such an extension is described for the purpose of practical numerical applications.

When the origin is not on S but lies on the same plane which contains S, then the numerical integration of Eq. (1) can be performed by suitably dividing the triangular area S. Three different cases are depicted in Fig. 6, according to the relative location of the origin to $S = \Delta ABC$. Let us denote $S_1 = \Delta OBC$, $S_2 = \Delta OAC$, and $S_3 = \Delta OAB$. The integral values which correspond to the areas S, S_1, S_2, and S_3 are denoted by I, I_1, I_2, and I_3, respectively. Note that we can effectively apply the present scaling integration method to compute I_1, I_2, and I_3, since the origin is shared as the vertex of each of the triangular area, S_1, S_2, and S_3.

Then, using the relative geometric relations among S, S_1, S_2, and S_3 (see Fig. 6), I is obtained as follows:

- Case 1.

$$I = I_1 + I_2 + I_3. \tag{14}$$

- Case 2.

$$I = I_2 + I_3 - I_1. \tag{15}$$

- Case 3.

$$I = I_1 - I_2 - I_3. \tag{16}$$

Thus, the numerical integration should be executed over the boundary line elements only along the edges **AB**, **BC**, and **CA** of ΔABC.

If the origin is not on the plane which includes S, then we may note that the numerical integration can be easily performed using the conventional methods [12], i.e., Newton or Gauss methods, except the case when the origin is located very close to S.

5. Conclusions

A new, accurate, and simple algorithm for the numerical evaluation of singular integrals is obtained using an inherent (analytical) scaling relationship between the integral value and triangular area of integration. The algorithm avoids the direct numerical evaluation of integrand in the neighborhood of singularities. Numerical advantages of the present scaling integration method are illustrated using the numerical examples whose analytical solutions are known. It is also demonstrated that the present method is useful and accurate in the BEM analyses. Although the method is most efficiently applied to the flat triangular elements, it is proved that the algorithm can improve both the computational time and numerical accuracy.

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